THE GROUP GENERATED BY GAMMA FUNCTIONS $\Gamma(ax+1)$, AND ITS SUBGROUP OF THE ELEMENTS CONVERGING TO CONSTANTS

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Abstract

Let G be the multiplicative group generated by the gamma functions $\Gamma(ax+1)$ ($a = 1, 2, \dots$), and H be the subgroup of all elements of G that converge to nonzero constants when $x \to \infty$. The quotient group G/H is the group of equivalence classes of G, where f and g are equivalent $\iff$ $f \sim Cg$ $(x \to \infty)$ for some $C \neq 0$. We show that $G/H \simeq \mathbb{Q}^+$. Similar consideration is possible for the case that the gamma functions $\Gamma(ax+1)$ with $a \in \mathbb{R}^+$ are concerned, and we show that $G/H \simeq \mathbb{Z} \times \mathbb{R} \times \mathbb{R}$.

Also, several concrete examples of the elements of H are constructed, e.g., it holds that $\frac{\binom{18n}{12n,3n,3n}}{\binom{18n}{9n,8n,n}} \longrightarrow \sqrt{\frac{2}{3}}$ $(n \to \infty)$, where $\binom{*}{*,\dots,*}$ denotes a multinomial coefficient.

1. Introduction

Throughout this paper, let $\mathbb{P}$ denote the set of all positive integers, $\mathbb{N}$ denote the set of all nonnegative integers, $\mathbb{Q}^+$ denote the multiplicative group of all positive rational numbers, and $\mathbb{R}^+$ denote the multiplicative group of all positive real numbers.

Let G be the multiplicative group generated by the gamma functions $\Gamma(ax+1)$ ($a \in \mathbb{P}$), and H be the subgroup of all elements of G that converge to nonzero constants when $x \to \infty$. By using the notation:

$$M^{a_1,\dots,a_s}_{b_1,\dots,b_t} = M^{a_1,\dots,a_s}_{b_1,\dots,b_t}(x) = \frac{\prod_{k=1}^{s} \Gamma(a_k x + 1)}{\prod_{k=1}^{t} \Gamma(b_k x + 1)}, \tag{1}$$

we have

$$G = \{ M^{a_1,\dots,a_s}_{b_1,\dots,b_t} \mid a_1, \dots, a_s, b_1, \dots, b_t \in \mathbb{P};\ s, t \in \mathbb{N} \}. \tag{2}$$

Here, when $st = 0$, $M^{a_1,\dots,a_s}_{b_1,\dots,b_t}$ becomes $M^{a_1,\dots,a_s}$, $M_{b_1,\dots,b_t}$, or $M = 1$.

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We consider the quotient group G/H, which is the group of equivalence classes of G, where f and g are defined to be equivalent when $f \sim Cg$ $(x \to \infty)$ for some nonzero constant C. We show the following.

Theorem 1. *It holds that* $G \simeq G/H \simeq \mathbb{Q}^+$.

Similar consideration is possible for the case that the gamma functions $\Gamma(ax+1)$ with $a \in \mathbb{R}^+$ are taken as the generators. Let $\tilde{G}$ be the multiplicative group generated by $\Gamma(ax+1)$ $(a \in \mathbb{R}^+)$, and $\tilde{H}$ be the subgroup of all elements of $\tilde{G}$ that converge to nonzero constants when $x \to \infty$.

Theorem 2. *It holds that* $\tilde{G}/\tilde{H} \simeq \mathbb{Z} \times \mathbb{R} \times \mathbb{R}$.

In Section 3, we study concrete elements of H in a combinatorial context. For partitions λ, μ, a primitive solution $(\lambda; \mu)$ to the condition for $M_\mu^\lambda \in H$ is defined, and we prove that there exists a primitive solution of length exceeding n for every integer n. Several concrete examples of the primitive solutions are also given.

2. Homomorphisms and proofs

First we show a limit lemma for the elements of G:

Lemma 1.

$$M_{b_1,\dots,b_t}^{a_1,\dots,a_s}(x) \sim \sqrt{\tfrac{a_1 \dots a_s}{b_1 \dots b_t}}\,(2\pi x)^{\frac{s-t}{2}} \left(\tfrac{x}{e}\right)^{x\left(\sum_{k=1}^s a_k - \sum_{k=1}^t b_k\right)} \left(\frac{a_1^{a_1} \dots a_s^{a_s}}{b_1^{b_1} \dots b_t^{b_t}}\right)^x. \tag{3}$$
$$(x \longrightarrow \infty)$$

Proof. This is a direct consequence of Stirling's formula: $\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x$. Put $x \to ax$ and calculate $M_{b_1,\dots,b_t}^{a_1,\dots,a_s}(x)$. □

By this lemma, we have the condition for the elements of G to be contained in H. Indeed, $M_{b_1,\dots,b_t}^{a_1,\dots,a_s}(x)$ converges to a nonzero constant if and only if the three factors $(2\pi x)^{\frac{s-t}{2}}$, $\left(\frac{x}{e}\right)^{x*}$, $(**{*})^x$ of (3) are constants equal to 1.

Lemma 2. *For* $M_{b_1,\dots,b_t}^{a_1,\dots,a_s} \in H$, *it is necessary and sufficient that (i)* $s = t$, *(ii)* $\sum_{k=1}^s a_k = \sum_{k=1}^t b_k$, *and (iii)* $a_1^{a_1} \dots a_s^{a_s} = b_1^{b_1} \dots b_t^{b_t}$.

Hereafter, we consider homomorphisms from G to certain groups for the preparation of a proof of Theorem 1. Let $\tilde{Q}$ denote the multiplicative group generated by $\{p^p \mid p :$ a prime$\}$. Let ϕ_i $(i = 1, 2, 3)$ be homomorphisms defined below:

$$\begin{aligned} \phi_1 &: G \longrightarrow \mathbb{Z} : M_{b_1,\dots,b_t}^{a_1,\dots,a_s} \longmapsto s - t \\ \phi_2 &: G \longrightarrow \mathbb{Z} : M_{b_1,\dots,b_t}^{a_1,\dots,a_s} \longmapsto \textstyle\sum_{k=1}^s a_k - \sum_{k=1}^t b_k \\ \phi_3 &: G \longrightarrow \tilde{Q} : M_{b_1,\dots,b_t}^{a_1,\dots,a_s} \longmapsto \frac{a_1^{a_1} \dots a_s^{a_s}}{b_1^{b_1} \dots b_t^{b_t}}. \end{aligned} \tag{4}$$

This definition is possible because each element of G is uniquely expressed by the symbol $M_{b_1,\ldots,b_t}^{a_1,\ldots,a_s}$ except a permutation of indices and a cancellation of identical upper/lower indices. Then we have a homomorphism:

$$\Phi : G \longrightarrow \mathbb{Z} \times \mathbb{Z} \times \tilde{Q} : g \longmapsto (\phi_1(g), \phi_2(g), \phi_3(g)). \tag{5}$$

Lemma 3. *Φ is a surjection.*

Proof. To begin with we note that ϕ_3 is well defined, say, $\phi_3(G) \subset \tilde{Q}$. Take any $g = M_{b_1,\ldots,b_t}^{a_1,\ldots,a_s} \in G$. If $a_1 = p^e m$ and $p \nmid m$ for a prime factor p, we have $a_1^{a_1} = (p^e m)^{p^e m} = p^{p(p^{e-1}em)} m^{a_1}$. Hence there exist prime numbers $p_1, \ldots, p_l$, and integers $e_1, \ldots, e_l$ such that

$$\frac{a_1^{a_1} \ldots a_s^{a_s}}{b_1^{b_1} \ldots b_t^{b_t}} = \prod_{k=1}^{l} p_k^{p_k e_k}, \tag{6}$$

and therefore $\phi_3(G) \subset \tilde{Q}$. Next we confirm that ϕ_3 is a surjection. For an arbitrary element $y = \frac{p_1^{p_1} \ldots p_s^{p_s}}{q_1^{q_1} \ldots q_t^{q_t}}$ of $\tilde{Q}$ with prime numbers $p_1, \ldots, p_s, q_1, \ldots, q_t$ (repetition allowed), taking $g = M_{q_1,\ldots,q_t}^{p_1,\ldots,p_s}$, we have $y = \phi_3(g)$.

Now, we prove Φ is a surjection. Let (d, l, y) be an arbitrary element of $\mathbb{Z} \times \mathbb{Z} \times \tilde{Q}$. Since ϕ_3 is a surjection, we have $y = \phi_3(g)$ for some $g = M_{b_1,\ldots,b_t}^{a_1,\ldots,a_s} \in G$. If $\phi_1(g) \neq d$, we can take $g_1 = M_{b_1,\ldots,b_t}^{a_1,\ldots,a_s,1,\ldots,1}$ or $M_{b_1,\ldots,b_t,1,\ldots,1}^{a_1,\ldots,a_s}$ such that $\phi_1(g_1) = d$, $\phi_3(g_1) = y$. If $\phi_2(g_1) = m \neq l$, consider $g_2 = M_{6,2,1}^{4,3,3}$. We see $\Phi(g_2) = (0, 1, 1)$. Thus, letting $g_3 = g_1 g_2^{l-m}$, we have $\Phi(g_3) = (d, l, y)$. □

Proof of Theorem 1. (i) $G \simeq \mathbb{Q}^+$: The mapping: $\psi : G \longrightarrow \mathbb{Q}^+$ defined by

$$\psi(M_{b_1,\ldots,b_t}^{a_1,\ldots,a_s}) = \frac{p_{a_1} \cdots p_{a_s}}{p_{b_1} \cdots p_{b_t}} \tag{7}$$

is confirmed to be an isomorphism, where p_i denotes the i-th prime.

(ii) $G/H \simeq \mathbb{Q}^+$: Apply the fundamental homomorphism theorem: $G/\ker\Phi \simeq \Phi(G)$ to the above-defined $\Phi : G \longrightarrow \mathbb{Z} \times \mathbb{Z} \times \tilde{Q}$. By definition of Φ and Lemma 2, $\ker\Phi = H$. Hence, together with Lemma 3, we have $G/H \simeq \mathbb{Z} \times \mathbb{Z} \times \tilde{Q}$.

There exist isomorphisms $i : \tilde{Q} \longrightarrow \mathbb{Q}^+$ and $j : \mathbb{Z} \times \mathbb{Q}^+ \longrightarrow \mathbb{Q}^+$ defined by $i(p^p) = p$ for every prime p and

$$j(n, 2^{n_2} 3^{n_3} 5^{n_5} \ldots) = 2^n 3^{n_2} 5^{n_3} \ldots. \tag{8}$$

Therefore

$$\mathbb{Z} \times \mathbb{Z} \times \tilde{Q} \simeq \mathbb{Z} \times \mathbb{Z} \times \mathbb{Q}^+ \simeq \mathbb{Q}^+. \tag{9}$$

This proves $G/H \simeq \mathbb{Q}^+$. □

Next, we extend the homomorphism Φ to $\tilde{\Phi}$ defined on $\tilde{G}$ in order to prove Theorem 2. For that purpose we extend the homomorphisms ϕ_i to $\tilde{\phi}_i$ defined on $\tilde{G}$ by the correspondence formulas used in (4). Then $\tilde{\Phi}$ is extended to a homomorphism:

$$\tilde{\Phi} : \tilde{G} \longrightarrow \mathbb{Z} \times \mathbb{R} \times \mathbb{R}^+ : g \longmapsto (\tilde{\phi}_1(g), \tilde{\phi}_2(g), \tilde{\phi}_3(g)). \tag{10}$$

Lemma 4. *$\tilde{\Phi}$ is a surjection.*

Proof. Let (d, x, y) be an arbitrary element of $\mathbb{Z} \times \mathbb{R} \times \mathbb{R}^+$. Clearly $\tilde{\phi}_3(g) = y$ for some $g \in \tilde{G}$. In a similar manner as in the proof of Lemma 3, we have $\tilde{\phi}_1(g_1) = d$ and $\tilde{\phi}_3(g_1) = y$ for some $g_1 \in \tilde{G}$. Now take $\eta \in (1/e^{1/e}, 1)$, then $x^x = \eta$ has distinct two real solutions $\theta_1, \theta_2 \in (0,1)$. Let $M_{\theta_1}^{\theta_2} = g(\eta)$, then $\tilde{\Phi}(g(\eta)) = (0, \theta_1 - \theta_2, 1)$, where $\theta_1 - \theta_2$ takes an arbitrary nonzero value in $(-1,1)$ depending on η. Hence choosing suitable $\eta_1, \ldots, \eta_l$, we have $g_2 = g_1 g(\eta_1) \ldots g(\eta_l)$ such that $\tilde{\Phi}(g_2) = (d, x, y)$. □

Proof of Theorem 2. Apply again the fundamental homomorphism theorem to $\tilde{\Phi}$. We have

$$\tilde{G}/\tilde{H} = \tilde{G}/\ker\tilde{\Phi} \simeq \tilde{\Phi}(\tilde{G}) = \mathbb{Z} \times \mathbb{R} \times \mathbb{R}^+ \simeq \mathbb{Z} \times \mathbb{R} \times \mathbb{R}. \tag{11}$$

□

3. Partitions and primitive solutions

In this section, we construct concrete examples of the elements of H, and study primitive solutions defined below corresponding to such elements $M_{b_1,\ldots,b_s}^{a_1,\ldots,a_s}(x)$ of H that generate H by ordinary multiplication and the variable transformation $x \longrightarrow kx$ for every positive integer k.

A partition of a positive integer n is a weakly decreasing sequence of positive integers: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_s)$ such that $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_s = n$. Each λ_i is called a part of λ and the integer s is called the length of λ denoted by $l(\lambda)$. Let $\lambda = (\lambda_1, \ldots, \lambda_s)$ and $\mu = (\mu_1, \ldots, \mu_t)$ be partitions of length s and t, respectively. Set $M_\mu^\lambda = M_{\mu_1,\ldots,\mu_t}^{\lambda_1,\ldots,\lambda_s}$, then any element of G is expressed in this form. Denote $\lambda^\lambda = \lambda_1^{\lambda_1} \ldots \lambda_s^{\lambda_s}$; $k\lambda = (k\lambda_1, \ldots, k\lambda_s)$ for a positive integer k; and denote by $\lambda \oplus \tilde{\lambda}$, the rearrangement of $(\lambda_1, \ldots, \lambda_s, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_{s'})$ in decreasing order. The condition for $M_\mu^\lambda \in H$ in Lemma 2 is rewritten as

$$\text{(i) } l(\lambda) = l(\mu) \qquad \text{(ii) } |\lambda| = |\mu| \qquad \text{(iii) } \lambda^\lambda = \mu^\mu. \tag{12}$$

Equations (12) have always solutions $\lambda = \mu$, which we call trivial solutions. For every solution $(\lambda; \mu)$ to (12), $l(\lambda) = l(\mu)$ and $|\lambda| = |\mu|$ are called the length and the size of the solution, respectively. The solutions $(\lambda; \mu)$ and $(\mu; \lambda)$ are usually identified. If $(\lambda; \mu)$ is a solution, then $k(\lambda; \mu) = (k\lambda; k\mu)$ is also a solution for every positive integer k, because

$$(k\lambda)^{k\lambda} = k^{k|\lambda|} \left(\lambda^\lambda\right)^k = k^{k|\mu|} \left(\mu^\mu\right)^k = (k\mu)^{k\mu}. \tag{13}$$

These solutions are called equivalent to each other. In addition, if two solutions $(\lambda; \mu)$, $(\tilde{\lambda}; \tilde{\mu})$ of positive lengths exist, then $(\lambda; \mu) \oplus (\tilde{\lambda}; \tilde{\mu}) = (\lambda \oplus \tilde{\lambda}; \mu \oplus \tilde{\mu})$ is also a solution, decomposable into two solutions. Hence it is important to find nontrivial solutions that can not be written in the form $k(\lambda; \mu)$ $(k \geq 2)$ nor $(\lambda; \mu) \oplus (\tilde{\lambda}; \tilde{\mu})$, which we call primitive solutions. It is easily seen that there are no nontrivial solutions of length ≤ 2.

Theorem 3. *For every positive integer n, there exists a primitive solution to* (12) *of length exceeding n.*

Proof. For convenience, we sometimes use the notation $\binom{\lambda}{\mu}$ for a solution $(\lambda;\mu)$ to (12). Also, we denote by $\{\lambda\}$ the multiset which consists of all parts of λ. We prove that the following is a primitive solution to (12) for $n \geq 8$:

$$\left(\begin{array}{cccccc} 2^n, & 2^{n-2}, 2^{n-2}, & \overbrace{2,2,\ldots,2}^{2^{n-2}}, & \overbrace{2,2,\ldots,2}^{2^{n-2}}, & \overbrace{2,2,\ldots,2}^{2^{n-2}} \\ 2^{n-1}, & 2^{n-1}, 2^{n-1}, & \underbrace{4,4,\ldots,4}_{2^{n-2}}, & \underbrace{1,1,\ldots,1}_{2^{n-2}}, & \underbrace{1,1,\ldots,1}_{2^{n-2}} \end{array} \right). \tag{14}$$

One can confirm that (14) is a solution of length $3 \times 2^{n-2} + 3$ and of size 3×2^n, and that $\lambda^\lambda = \mu^\mu = 2^{2^{n-1}(3n+1)}$. It is also clear that (14) is not a k times multiple of some solution for $k \geq 2$. Thus it suffices to show that (14) is not decomposable into two solutions.

Suppose (14) is decomposed into $(\sigma;\tau) \oplus (\tilde{\sigma};\tilde{\tau})$. Write (14) as $\binom{\lambda}{\mu} = \binom{\lambda^1,\lambda^2}{\mu^1,\mu^2}$, where λ^1 is the first three parts of λ, λ^2 is the rest of it, and μ^1, μ^2 are defined similarly. If σ and τ are composed by choosing only the parts of λ^2 and μ^2, respectively, then from $|\sigma| = |\tau|$, it follows that $\binom{\sigma}{\tau}$ is consist of the blocks $\binom{2,2,2}{4,1,1}$, which contradicts $\sigma^\sigma = \tau^\tau$. For the case that σ and τ contain only some parts of λ^1 and μ^2 (or λ^2 and μ^1), respectively, the only possibility is $n = 2, 4$ ($n = 2$). Also, it is clearly impossible that σ and τ could contain only some parts of λ^1 and μ^1, respectively. Thus we should deal with the case that σ contains both parts of λ^1 and λ^2 or τ contains both parts of μ^1 and μ^2. If σ contains all parts of λ^1 and τ contains no parts of μ^1, we have

$$\frac{\sigma^\sigma}{\tau^\tau} \geq \frac{2^{n2^n} 2^{(n-2)2^{n-2}\times 2}}{2^{8\times 2^{n-2}}} = 2^{(3n-6)2^{n-1}} > 1 \tag{15}$$

for $n \geq 3$. The alternative case that σ contains no parts of λ^1 and τ contains all parts of μ^1 is very similar. Hence we consider the case that σ or τ has a nonempty proper submultiset of $\{\lambda^1\}$ or $\{\mu^1\}$ as parts, respectively. (The other cases already appear above for $(\sigma;\tau)$ or $(\tilde{\sigma};\tilde{\tau})$.) If $\{\sigma\} \cap \{\lambda^1\} = \{2^{n-2}\}$ and $\{\tau\} \cap \{\mu^1\} = \emptyset$ (as multiset), then

$$\frac{\sigma^\sigma}{\tau^\tau} \geq \frac{2^{(n-2)2^{n-2}} 2^{2\times(2^{n-2}-1)}}{2^{8\times 2^{n-2}}} = 2^{(n-8)2^{n-2}-2}. \tag{16}$$

Hence for $n \geq 9$, $\frac{\sigma^\sigma}{\tau^\tau} > 1$, and so $(\sigma;\tau)$ is not a solution. For $n = 8$, the only possibility that fits $\frac{\sigma^\sigma}{\tau^\tau} = 1$ is

$$(2^{n-2}, \underbrace{2,2,\ldots,2}_{2^{n-2}}; \underbrace{4,4,\ldots,4}_{2^{n-2}}, 1), \tag{17}$$

but $|\sigma| \neq |\tau|$. Therefore, for $n \geq 8$, $(\sigma;\tau)$ is not a solution.

If $\{\sigma\} \cap \{\lambda^1\} = \{2^{n-2}\}$ and $\{\tau\} \cap \{\mu^1\} = \{2^{n-1}\}$, then

$$\frac{\sigma^\sigma}{\tau^\tau} \leq \frac{2^{(n-2)2^{n-2}} 2^{2\times 2^{n-2}\times 3}}{2^{(n-1)2^{n-1}}} = 2^{(6-n)2^{n-2}}. \tag{18}$$

Hence for $n \geq 7$, $(\sigma;\tau)$ is not a solution.

Although we can proceed in the similar manner to the goal, we give one more case $\{\sigma\} \cap \{\lambda^1\} = \{2^{n-2}, 2^{n-2}\}$ and $\{\tau\} \cap \{\mu^1\} = \{2^{n-1}\}$ that has a little different flavor. Let σ^2 be the partition consists of the parts of σ contained in λ^2, and τ^2 be defined similarly. Since $2^{n-2} + 2^{n-2} = 2^{n-1}$, we have $|\sigma^2| = |\tau^2|$. As $\frac{2^{(n-2)2^{n-2}} 2^{(n-2)2^{n-2}}}{2^{(n-1)2^{n-1}}} = 2^{-2^{n-1}} < 1$,

τ contains 1 parts, and by parity, at least two 1 parts. For the partition $\tilde{\sigma}^2$ obtained by exclusion of a 2 part from σ^2, and the partition $\tilde{\tau}^2$ obtained by exclusion of two 1 parts from τ^2, we have $|\tilde{\sigma}^2| = |\tilde{\tau}^2|$ and $l(\tilde{\sigma}^2) = l(\tilde{\tau}^2)$. Hence $\binom{\tilde{\sigma}^2}{\tilde{\tau}^2}$ is composed of the blocks $\binom{2,2,2}{4,1,1}$, and therefore

$$\frac{\sigma^\sigma}{\tau^\tau} = 2^{-2^{n-1}} 2^{-2s} 2^2 = 2^{-2^{n-1}-2(s-1)} = 1. \tag{19}$$

This fails for all $n \geq 3$. □

The solution (14) is primitive also for $n = 6, 7$, which is confirmed by showing the equations:

$$\begin{cases} a + b + c = d + e + f \\ 2^n a + 2^{n-2} b + 2c = 2^{n-1} d + 4e + f \\ 2^{n 2^n a} 2^{(n-2) 2^{n-2} b} 2^{2c} = 2^{(n-1) 2^{n-1} d} 2^{8e} \end{cases} \tag{20}$$

have no solution (a, b, c, d, e, f) of nonnegative integers with $a \leq 1$, $b \leq 2$, $c \leq 3 \times 2^{n-2}$, $d \leq 3$, $e \leq 2^{n-2}$, $f \leq 2^{n-1}$ for $n = 6, 7$, except 0 or $(1, 2, 3 \times 2^{n-2}, 3, 2^{n-2}, 2^{n-1})$. However, for $n = 5$, (14) is decomposed into:

$$\begin{pmatrix} 8,2,2,2,2,2,2,2,2 \\ 4,4,4,4,4,1,1,1,1 \end{pmatrix} \oplus \begin{pmatrix} 32,8,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2 \\ 16,16,16,4,4,4,1,1,1,1,1,1,1,1,1,1,1,1 \end{pmatrix}. \tag{21}$$

By computational calculation, many primitive solutions of length ≥ 4 are easily found (several examples are listed below). However, no solutions of length 3 are found except ones equivalent to $(12, 3, 3; 9, 8, 1)$ for the size ≤ 2000.

length	primitive solutions
3	$\binom{12,3,3}{9,8,1}$
4	$\binom{9,4,4,2}{6,6,6,1}$
5	$\binom{10,4,2,2,2}{8,5,5,1,1}$, $\binom{8,3,3,3,3}{6,6,4,2,2}$, $\binom{16,6,3,3,2}{12,8,8,1,1}$, $\binom{14,6,4,3,3}{12,7,7,2,2}$, $\binom{12,5,5,4,4}{10,8,6,3,3}$
6	$\binom{6,2,2,2,2,2}{4,4,3,3,1,1}$, $\binom{8,3,3,2,2,2}{6,4,4,4,1,1}$, $\binom{10,3,3,3,3,2}{6,6,5,5,1,1}$, $\binom{12,5,5,2,2,2}{10,6,6,4,1,1}$, $\binom{12,4,4,3,3,2}{8,6,6,6,1,1}$, $\binom{10,6,3,3,3,3}{9,5,5,4,4,1}$, $\binom{10,4,4,4,3,3}{8,6,5,5,2,2}$, $\binom{10,9,4,2,2,2}{12,5,5,3,3,1}$
7	$\binom{9,2,2,2,2,2,2}{6,6,3,3,1,1,1}$, $\binom{12,2,2,2,2,2,2}{8,6,6,1,1,1,1}$, $\binom{9,4,4,4,2,2,2}{8,6,3,3,3,3,1}$, $\binom{15,3,2,2,2,2,2}{10,9,5,1,1,1,1}$, $\binom{12,6,2,2,2,2,2}{9,8,4,4,1,1,1}$, $\binom{12,4,4,4,2,2,2}{8,8,6,3,3,1,1}$, $\binom{12,3,3,3,3,3,3}{9,6,6,4,2,2,1}$, $\binom{9,4,4,4,4,4,1}{8,6,6,3,3,2,2}$
8	$\binom{10,3,3,3,3,3,3,1}{9,5,5,2,2,2,2,2}$, $\binom{12,3,3,3,3,2,2,2}{9,6,4,4,4,1,1,1}$
9	$\binom{8,2,2,2,2,2,2,2,2}{4,4,4,4,4,1,1,1,1}$, $\binom{10,3,3,2,2,2,2,2,2}{6,5,5,4,4,1,1,1,1}$
10	$\binom{9,4,2,2,2,2,2,2,2,2}{8,3,3,3,3,3,3,1,1,1}$

TABLE 1. Primitive solutions of length ≤ 10 and size ≤ 30

Conjecture 1. *A primitive solution to* (12) *of length 3 is unique.*

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